JOURNAL OF APPROXIMATION THEORY **49**, 378–389 (1987)

Optimal Interpolation with Polynomials Having Fixed Roots

THEODORE KILGORE

Division of Mathematics, Department of Algebra, Combinatorics, and Analysis, Auburn University, Alabama 36849, U.S.A.

Communicated by R. Bojanic

Received November 4, 1984; revised October 13, 1985

INTRODUCTION

Let Y be the space of polynomials of degree $n+m$ or less where $n \geqslant 1$, $m \geqslant 0$, and $n+m \geqslant 2$,with roots at the given points $t_{n+1},..., t_{n+m}$. If nodes of interpolation $t_0,..., t_n$ are chosen such that

$$t_0 < t_1 < \cdots < t_n,$$

and such that

$$t_{n+k} \notin [t_0, t_n] \text{ for } k \in \{1,..., m\},$$

it is possible to construct in Y *fundamental polynomials* $y_0,..., y_n$ such that

$$y_i(t_j) = \delta_{ij} \text{ (Kronecker delta)}$$
$$\text{for } i \in \{0,..., n\} \text{ and for } j \in \{0,..., n\}.$$

If $[a, b]$ is any interval containing $[t_0, t_n]$, it is then possible to construct an interpolating projection

$$L: C[a, b) \to Y$$

by defining

$$Lf = \sum_{i=0}^{n} f(t_i)\, y_i \qquad \text{for} \quad f \in C[a, b].$$

Clearly, L is bounded, and

$$\| L \| = \left\| \sum_{i=0}^{n} | y_i | \right\|.$$

0021-9045/87 $3.00

Our purpose here is to minimize $\| L \|$ for various choices of the interval $[a, b]$ and for various constraints upon the points $t_0,..., t_{n+m}$, noting that the positioning and length of the interval $[t_0, t_{n+m}]$ will not affect the value of $\| L \|$, since an affine transformation of the interval will give rise to a new interpolation with the same norm. Hence, t_0 and t_{n+m} will usually be assumed to be fixed but arbitrary points.

Notation. For notational convenience, we will begin by assuming that the roots $t_{n+1},..., t_{n+m}$ lie to the right of t_n in the order

$$t_n < t_{n+1} < \cdots < t_{n+m}.$$

We define, for $i \in \{1,..., n\}$, X_i to be the polynomial (in Y) which agrees with $\sum_{j=0}^{n} | y_i |$ on the interval $[t_{i-1}, t_i]$, and we define

$$X_{n+1} = \cdots = X_{n+m} = \sum_{j=0}^{n} (-1)^j y_j,$$

noting that

$$| X_{n+k} | = \sum_{j=0}^{n} | y_j | \qquad \text{on} \quad [t_{n+k-1}, t_{n+k}]$$

and

$$X_{n+k}(t_{n+1}) = \cdots = X_{n+k}(t_{n+m}) = 0, \qquad \text{for} \quad k \in \{1,..., m\}.$$

We also define, for $i \in \{1,..., n+m\}$, $i \neq n+1$,

$$\lambda_i = \max_{[t_{i-1}, t_i]} \sum_{j=0}^{n} | y_j |,$$

and T_i as the point in (t_{i-1}, t_i) at which λ_i is attained. We note that

$$X_i(T_i) = \lambda_i$$

and

$$X_i'(T_i) = 0, \qquad \text{for} \quad i \in \{1,..., n+m\}, i \neq n+1.$$

The seemingly exceptional λ_{n+1} and T_{n+1} are defined in essentially the same manner, by stating that T_{n+1} is the nearest root of X_{n+1} immediately to the left of t_{n+1}, and that

$$\lambda_{n+1} = | X_{n+1}(T_{n+1}) |.$$

Thus, the potential problem that

$$\max_{[t_n, t_{n+1}]} |X_{n+1}|$$

might occur at t_n is avoided, and

$$X'_{n+1}(T_{n+1}) = 0$$

holds.

Results

THEOREM 1. *If interpolation is carried out on the interval $[a, b]$ with polynomials having nodes of interpolation $t_0, \ldots, t_n$ and roots at $t_{n+1}, \ldots, t_{n+m}$ such that $a = t_0 < \cdots < t_n < t_{n+1} < \cdots < t_{n+m} = b$, then the interpolation of minimal norm obeys the "Bernstein condition" [1] that*

$$\lambda_1 = \cdots = \lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m},$$

and the system of points $t_1, \ldots, t_{n+m-1}$ which yields this equality is unique. The quantities $\lambda_1, \ldots, \lambda_{n+m}$ also obey the "Erdös condition" [3] that, if one of them is greater than the common value stated above to characterize optimal interpolation, another of them is less.

Remark. What Theorem 1 does not do is to allow the node t_n and the roots $t_{n+1}, \ldots, t_{n+m}$ to be fixed. Such pre-positioning of these points leads to a problem of greater complexity, for which the following results may be stated.

THEOREM 2. *If interpolation is done on an interval $[a, b]$ with polynomials having fixed roots $t_{n+1}, \ldots, t_{n+m}$ and nodes of interpolation $t_0, \ldots, t_n$, such that*

$$a = t_0 < \cdots < t_n = b < t_{n+1} < \cdots < t_{n+m},$$

then

(i) *interpolation of minimal norm is characterized by the Bernstein condition that*

$$\lambda_1 = \cdots = \lambda_n,$$

which is produced by a unique choice of nodes.

(ii) *the quantities*

$$\lambda_1, \ldots, \lambda_n$$

obey the Erdös condition that if one of them is greater than the common value given in (i), *another is less.*

(iii) *the norm of interpolation is governed by the ratio*

$$\frac{b-a}{t_{n+1}-b}.$$

Specifically, the norm increases without bound as $b \to t_{n+1}$ *and decreases as* $b \to a$, *with lower limit equal to the norm of optimal Lagrange interpolation with polynomials of degree n or less.*

COROLLARY 1. *If* t_0 *and* $t_{n+1},..., t_{n+m}$ *are fixed and if* $a = t_0$ *and* $b = t_{n+m}$, *then optimal interpolation on* $[a, b]$ *is characterized by the condition that*

$$\lambda_1 = \cdots = \lambda_n = \max\{\lambda_{n+1},..., \lambda_{n+m}\},$$

which occurs at a unique placement of the nodes

$$t_0,..., t_n,$$

and the Erdös condition also holds on the given maxima.

COROLLARY 2. *Theorem* 2 *and Corollary* 1 *also hold when the space of interpolation consists of all multiples of the function*

$$(t-t_{n+1})^{k_1} \cdots (t-t_{n+m})^{k_m}$$

by a polynomial of degree n or less, with

$$k_j > 1 \qquad \text{for} \quad j \in \{1,..., n\}.$$

COROLLARY 3. *Some or all of the roots*

$$t_{n+1},..., t_{n+m}$$

can be to the left of t_0 *as well as to the right of* t_n, *and the above results are still valid.*

A SKETCH OF THE PROOFS

For Theorem 1, the presence of additional fundamental polynomials is necessary. We define

$$y_{n+1},..., y_{n+m}$$

in like manner to the definition of $y_0,..., y_n$. The following discussion, devoted specifically to the proof of Theorem 1, will also serve as a preliminary to discussion of the remaining results.

Proof of Theorem 1. One notes that the functions

$$\partial\lambda_i/\partial t_j = -y_j(T_i\, X_i'(t_j),\ i\in\{1,\ldots,n+m\},\ j\in\{0,\ldots,n+m\}$$

exist and are continuous in $t_0,\ldots,t_{n+m}$. The points $T_0,\ldots,T_{n+m}$, of course, depend in an analytic fashion upon the nodes.

All of our results will follow from properties of various submatrices of

$$A=(\partial\lambda_i/\partial t_j)_{i=1}^{n+m}{}_{j=1}^{n+m-1},$$

which represents the derivative of the function

$$(t_1,\ldots,t_{n+m-1})\to(\lambda_1,\ldots,\lambda_{n+m}).$$

We define A_i for $i\in\{1,\ldots,n+m\}$ to be the matrix obtained by deleting the ith column of A.

To prove Theorem 1, it suffices to show

(1) det $A_i\neq 0$ for $i\in\{1,\ldots,n+m\}$ for arbitrary $t_0,\ldots,t_{n+m}$, and

(2) det A_i alternates in sign on $\{1,\ldots,n+m\}$.

To show (1) and (2), we first perform some row and column cancellations.

For $j\in\{1,\ldots,n+m-1\}$, the jth row of A is given by

$$-y_j(T_1)\,X_1'(t_j)\cdots -y_j(T_n)\,X_n'(t_j).$$

It is possible therefore to multiply the jth row by the denominator of y_j, namely by

$$\prod_{\substack{l=0\\ l\neq j}}^{n+m}(t_j-t_l).$$

When this procedure has been completed, the ith column, for $i\in\{1,\ldots,n+m\}$ is of the form

$$\prod_{\substack{j=0\\ j\neq 1}}^{n+m}(T_i-t_j)\,X_i'(t_1)$$

$$\cdots$$

$$\prod_{\substack{j=0\\ j\neq n+m-1}}^{n+m}(T_i-t_j)\,X_i'(t_{n+m-1}),$$

and the non-zero quantity

$$\prod_{j=0}^{n+m} (T_i - t_j).$$

may be divided from the ith column, leaving the matrix in the form

$$B = \begin{pmatrix} \dfrac{X_1'(t_1)}{t_1 - T_1} & \cdots & \dfrac{X_{n+m}'(t_1)}{t_1 - T_{n+m}} \\ \cdots & \cdots & \cdots \\ \dfrac{X_1'(t_{n+m-1})}{t_{n+m-1} - T_1} & \cdots & \dfrac{X_{n+m}'(t_{n+m-1})}{t_{n+m-1} - T_{n+m}} \end{pmatrix}$$

in which the expression

$$q_i(t) = \frac{X_i'(t)}{t - T_i}, \qquad i \in \{1, ..., n+m\}$$

is a polynomial of degree $n+m-2$ or less which is evaluated at the successive points $t_1, ..., t_{n+m-1}$ down the ith column of the matrix.

Conditions (1) and (2) now follow from the fact that, for all $p \in \{1, ..., n+m\}$,

$$\{q_1, ..., q_{n+m}\} \setminus \{q_p\}$$

is a linearly independent set. To establish this linear independence, we note first that the polynomials

$$q_1, ..., q_{n+m}$$

obey the following sign changes on the points

$$T_1, ..., T_{n+m}.$$

Assuming that the polynomials are all nonnegative at T_1 as a regularizing assumption, we have

(a) $q_j(T_1) > 0$ for $j \in \{1, ..., n+m\}$

(b) $\operatorname{sgn} q_i(T_i) = (-1)^i$ for $i \in \{2, ..., n+m\}$

(c) $\operatorname{sgn} q_1(T_i) = (-1)^i$ for $i \in \{2, ..., n+m\}$

(d) $q_j(T_i)\, q_1(T_i) \leqslant 0$ for $i, j \in \{1, ..., n+m\}$, $j \neq i$, and the inequality is strict unless both $i > n$ and $j > n$.

We now state the following

PROPOSITION 1. *Let $q_1,\ldots, q_{n+m}$ be polynomials whose degree is not more than $n+m-2$, satisfying properties* (a) *through* (d). *Then, for any $p \in \{1,\ldots, n+m\}$, the set*

$$\{q_1,\ldots, q_{n+m}\}\backslash\{q_p\}$$

is linearly independent.

Proof of Proposition 1. We assume that a linear combination

$$\sum_{j=1}^{n+m} a_j q_j = 0,\ a_p = 0 \text{ for some } p \in \{1,\ldots, n+m\}$$

has been given, in which we may also assume that a_1 is nonnegative. We proceed to show that all coefficients must be zero.

To this end, we define

$$\mathscr{S} = \{j : j \neq 1 \text{ and } a_j \geqq 0\}$$

and

$$\mathscr{R} = \{1,\ldots, n+m\}\backslash\mathscr{S}$$

and

$$R = \sum_{j \in \mathscr{R}} a_j q_j$$

and

$$S = \sum_{j \in \mathscr{S}} a_j q_j.$$

We note that $\mathscr{R} \neq \varnothing$, since $1 \in \mathscr{R}$. If $p \neq 1$, it follows immediately that $\mathscr{S} \neq \varnothing$ because $p \in \mathscr{S}$. If, however, $p = 1$, then $a_1 = 0$. It necessarily follows by (a) that either $a_j = 0$ for all $j \in \{1,\ldots, n+m\}$, in which case no further proof is needed, or else there are some indices for which the coefficient a_j is negative and others for which it it is positive. Thus in any case $\mathscr{R}$ and $\mathscr{S}$ are both nonempty, and we have

(e) $\qquad S + R = 0, \qquad S(T_1) \geqq 0, \text{ and } R(T_1) \leqq 0,$

with both inequalities strict if $a_j > 0$ for some $j \in \mathscr{S}$.

If $i \in \mathscr{S}$, then by (d),

$$q_j(T_i)\, q_1(T_i) \leqq 0$$

for all $j \in \mathscr{R}$, $j \neq 1$, whence, since $a_1 \geqq 0$,

$$a_j q_j(T_i)\, q_1(T_i) \geqq 0 \qquad \text{for all} \quad j \in \mathscr{R}.$$

Thus

(f) $R(T_i)\, q_1(T_i) \geqq 0$ whenever $i \in \mathscr{S}$.

If $i \in \mathscr{R}$, $i \neq 1$, then by (d) again

$$q_j(T_i)\, q_1(T_i) \leqq 0 \qquad \text{for} \quad j \in \mathscr{S},$$

whence

$$a_j q_j(T_i)\, q_1(T_i) \leqq 0 \qquad \text{for} \quad j \in \mathscr{S}.$$

Thus

(g) $R(T_i)\, q_1(T_i) = -S(T_i)\, q_1(T_i) \geqq 0$ whenever $i \in \mathscr{R}$ and $i \neq 1$. By (c), (e), (f), and (g), therefore,

$$(-1)^i R(T_i) \geqq 0 \qquad \text{for} \quad i \in \{1,..., n+m\},$$

whence $R = 0$ and $S = 0$. But if $S = 0$, then, as noted in (e), $a_j = 0$ for all $j \in \mathscr{S}$, for otherwise $S(T_1) > 0$. Hence,

$$a_j \leqq 0 \qquad \text{for all} \quad j \in \{2,..., n+m\}.$$

If now $p = 1$, we have $a_1 = 0$, and $R(T_1) = 0$ implies that all of the coefficients are zero.

If $p \neq 1$, then by (d),

$$q_j(T_p)\, q_1(T_p) \leqq 0 \qquad \text{for} \quad j \in \{2,..., n+m\},$$

and

$$a_1 > 0 \qquad \text{would imply} \quad R(T_p) > 0.$$

Since $R = 0$ is already established, this in turn implies that

$$a_1 = 0.$$

Under these conditions, we note that, since $a_1 = 0$, all other coefficients are also zero because of the fact that $R(T_1) = 0$.

Since all coefficients have been shown equal to zero, the linear independence of

$$\{q_1,..., q_{n+m}\} \setminus \{q_p\}$$

has been demonstrated.

Remaining details of the proof of Theorem 1 parallel closely the arguments used in [5, 6] to demonstrate similar results and will be omitted here.

Proof of Theorem 2. For the proof of Theorem 2, it is necessary to refine the arguments previously used. Specifically, the following new result is needed. Upon its demonstration, parts (i) and (ii) of Theorem 2 follow immediately, in the manner of Theorem 1.

PROPOSITION 2. *Let polynomials*

$$q_1, \ldots, q_{n+m}$$

and points

$$T_1, \ldots, T_{n+m}$$

satisfy the hypotheses of Proposition 1, *and let points*

$$t_1, \ldots, t_{n-1}$$

be situated so that

$$T_1 < t_1 < T_2 < \cdots < T_{n-1} < t_{n-1} < T_n.$$

Then, for $k \in \{1, \ldots, n\}$,

$$\det(q_i(t_j))_{\substack{i=1, j=1 \\ i \neq k}}^{n, n-1} \neq 0.$$

Proof of Proposition 2. If $m = 0$, this result rephrases Proposition 1. We use induction on m. Assuming that $m > 0$, choose points $t_n, \ldots, t_{n+m-2}$, such that

$$T_n < t_n < T_{n+1} < \cdots < T_{n+m-2} < t_{n+m-2} < T_{n+m-1}$$

and let

$$r(t) = \prod_{j=1}^{n+m-2} \frac{(t - t_j)}{(T_n - t_j)}.$$

Clearly, r alternates sign on the points

$$T_1, \ldots, T_{n+m-1},$$

while, for $i \in \{1, \ldots, n\}$ and for $j \in \{1, \ldots, n+m\}$,

$$\operatorname{sgn} q_i(T_j) = \begin{cases} \operatorname{sgn} q_i(T_i) & \text{if } \ j = i-1, i, \text{ or } i+1 \\ (-1)^{i+j+1} \operatorname{sgn} q_i(T_i) & \text{otherwise.} \end{cases}$$

Therefore, for $i \in \{1, \ldots, n\}$, the polynomial

$$-q_i(T_{n+m})r$$

agrees in sign with q_i on the points

$$T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{n+m-1}$$

while disagreeing in sign on T_i and T_{n+m}.
Thus,

$$q_i - q_i(T_{n+m})r$$

is clearly not the zero polynomial and, being of degree $n+m-2$ or less, must agree in sign with q_i itself on the points

$$T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{n+m-1}.$$

Since

$$[q_i - q_i(T_{n+m})r](T_{n+m}) = 0,$$

it must also agree in sign with q_i at the point T_i, for otherwise it would have at least $n+m-1$ roots.

Moreover,

$$q_i(t_j) = [q_i - q_i(T_{n+m})r](t_j) \qquad \text{for} \quad j \in \{1, \ldots, n-1\}.$$

Thus,

$$\det(q_i(t_j))_{\substack{i=1,\, j=1 \\ i \neq k}}^{n,n-1} = \det([q_i - q_i(T_{n+m})r](t_j))_{\substack{i=1,\, j=1 \\ i \neq k}}^{n,n-1} \quad \text{for each } k \in \{1, \ldots, n-1\}.$$

But the matrix on the right has a common factor of

$$t_j - T_{n+m}$$

across the jth row, for $j \in \{1, \ldots, n+m-1\}$. The matrix which remains after cancellation of these factors has entries which consist of polynomials of degree $n+m-3$ or less, which must satisfy the original hypotheses on a set of points

$$T_1, \ldots, T_{n+m-1},$$

and m has been reduced to $m-1$.

Thus, by the induction hypothesis, the matrix on the right has a nonzero determinant.

This completes the proof of Proposition 2, which as mentioned, supports the proof of Theorem 2, parts (i) and (ii).

To demonstrate (iii) of Theorem 2, it suffices to note that non-singularity conditions demonstrated in the above proposition imply that the nodes

$$t_1, ..., t_{n-1}$$

may be varied as an implicit function of t_n, under the condition that

$$\lambda_1, ..., \lambda_{n-1}$$

retain any given set of initial values, for $t_n \in (t_0, t_{n+1})$. Specifically, the obvious analogues of conditions (1) and (2) stated in the proof of Theorem 1 imply the existence of such a function. In order that the argument be completed, it is necessary that this implicit function be globally defined, but this problem has also been handled in analogous circumstances [5, 6]. Now, as $t_n \to t_0$, it is necessary that $\lambda_{n+1} \to \infty$, and, at the same time, λ_n must decrease. Moreover,

$$\partial \lambda_{n+1} / \partial t_n < 0$$

and

$$\partial t_n / \partial t_n > 0.$$

Meanwhile, as $t_n \to t_0$, the fundamental polynomials $y_0, ..., y_n$ uniformly and smoothly approach their counterparts of degree n on the shrinking interval $[t_0, t_n]$.

Clearly, (iii) of Theorem 2 follows.

The above arguments also imply Corollary 1.

To demonstrate Corollary 2, it suffices to note that the matrix cancellations described above in the proof of Theorem 1 are still applicable. Specifically, if we define

$$y_j(t) = (t - t_{n+1})^{k_1} \cdots (t - t_{n+m})^{k_m} \prod_{\substack{l=0 \\ l \neq j}}^{n} \frac{t - t_l}{t_j - t_l},$$

the formula

$$\partial \lambda_i / \partial t_j = -y_j(T_i)\, X_i'(t_j), \qquad i, j \in \{1, ..., n\}$$

remains valid, and therefore the reduction

$$(\partial \lambda_i / \partial t_j)_{i=1, j=1}^{n,n} = \left(\frac{X_i'(t_j)}{t_j - T_i} \right)_{i=1, j=1}^{n,n}$$

may be carried out as before, with the matrix on the right being an evaluation matrix. At this state, it is possible further to cancel from the jth row the factor

$$(t_j - t_{n+1})^{k_1 - 1} \cdots (t_j - t_{n+m})^{k_m - 1},$$

for $j \in \{1, ..., n\}$, and the remaining matrix is an evaluation matrix of polynomials with roots interlacing as before, satisfying the hypotheses of Proposition 2. Thus, the proofs of Theorem 2 and of Corollary 1 may be repeated in this context.

Corollary 3 is clear.

Concluding Remarks

This article continues the program laid down in [6] of extending the application of the Bernstein [1] and Erdös [3] conjectures on optimal Lagrange interpolation, upheld in [4, 5, 2], to a wider class of spaces. Proposition 2, new in this article, overcomes one of the difficulties mentioned in [6]. This article complements and extends the results of [6].

Bibliography

1. S. Bernstein, Sûr la limitation des valeurs d'un polynome $P_n(x)$ de degré n sûr tout un segment par ses valeurs en $n+1$ points du segment, *Izv. Akad. Nauk. SSSR* **7** (1931), 1025–1050.
2. C. De Boor and A. Pinkus, Proof of the conjectures of Bernstein and Erdös concerning polynomial interpolation, *J. Approx. Theory* **24** (1978), 289–303.
3. P. Erdös, Some remarks on polynomials, *Bull. Amer. Math. Soc.* **53** (1947), 1169–1176.
4. T. A. Kilgore, Optimization of the Lagrange interpolation operator, *Bull. Amer. Math. Soc.* **83** (1977), 1069–1071.
5. T. A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, *J. Approx. Theory* **24** (1978), 273–288.
6. T. A. Kilgore, Optimal interpolation with incomplete polynomials, *J. Approx. Theory* **41** (1984), 279–290.
7. T. A. Kilgore, A note on functions with interlacing roots, *J. Approx. Theory* **43** (1985), 25–28.